On classical properties of thermal squeezed states

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Abstract. We use the formalism of thermo field dynamics and the classical Cramér's theorem to show that if two quantum systems are prepared independently and their centre of mass is found to be in a thermal squeezed state, then both systems were prepared in thermal squeezed states. This constitutes an alternative derivation of the quantum version of Cramér's theorem. Further, we consider the case of two systems that are prepared independently and their centre of mass is in a pure state in the expanded space of thermo field dynamics. In this case we show that the system is also separable in the centre of mass and relative coordinates, and again all the states involved are thermal squeezed states.

1. Introduction

The analogue of classical behaviour in quantum systems is usually of great interest. Thus coherent states, ever since Schrödinger's early work [1] and, later, Glauber's work [2], have been characterized as being as near to classical states as quantum mechanics allows. More than twenty years ago, Aharonov *et al* (AFLP) [3] characterized a coherent state by an apparently different 'classical' requirement. They considered the indistinguishability of the radiation in two separate channels, whether it has been produced by independent sources or by a single source whose output is divided between the channels. Classically these two possibilities cannot be distinguished. They proved that the unique *quantum* state for which there is no distinction between the two cases is the pure coherent state. More recently, new classical properties of quantum coherent states have been discussed by Emch and Hegerfeldt (EH) [4]. Some of the relationships between their work and that of AFLP have been discussed in [5]. There the concept of thermal coherent state (TCS) was considered within the framework of thermo field dynamics (TFD) [6]. In the present work we study a corresponding classical property of thermal squeezed states (TSS) [7]. (For a recent review of (zero-temperature) squeezed states, see, e.g., [8]. For a discussion of possible representations of TSS and their physical meaning, see, e.g., [9]. An experiment involving the squeezing of thermal radiation is described in [10].)

The particular classical property that we discuss is the following: if two quantum systems are prepared independently, and if their centre of mass (CM) is found to be in a TSS, then each of the component systems is also in a TSS. In fact, even if the centre of mass is found to be in a TCS, each of the component systems will in general be in a TSS. In our proof we use only the classical version of Cramér's theorem [11].

Alternatively, this constitutes a different proof of the quantum version of the classical Cramér theorem [4]. In the course of the proof it will be obvious that the term 'centre of mass' may really be replaced by any *non-trivial* linear combination of the variables of the two systems. The use of TFD allows a simple delineation of two physically distinct cases: one wherein the density matrix of the two systems can be factorized in both the individual systems' coordinates and in the systems' CM-relative coordinates, and the other where only the former factorization is possible (i.e. the systems are prepared independently but the CM and relative coordinates are not independent).

The organization of the paper is as follows. In section 2 we discuss characteristic functions (CF) and the extension of Cramér's theorem to quantum systems. In section 3 we consider the case where the CM of the two independent systems is known to be in a pure state in TFD. We show that this suffices to determine that the density matrix of the whole system factorizes not only in the original particles' coordinates but also in the CM-relative coordinates, i.e. it is bifactorizable. Hence [12] all density matrices involved have the form of a general Gaussian in the coordinate representation. A general Gaussian density matrix yields a general Gaussian CF. It was pointed out in [7] and shown explicitly in [13] that a Gaussian CF is equivalent to a TSS. Henceforth we shall use the terms TSS and Gaussian density matrix interchangeably. The last section summarizes the results of this paper.

2. Characteristic functions and Cramér's theorem

The tools of our analysis in this paper are provided by TFD. Ample reviews of TFD are given in [6, 14]. In TFD, every degree of freedom is doubled: each original physical operator has its own counter-partner, called a tilde operator, according to the tilde conjugation rules [6, 14]. Thus the whole state vector space in TFD, $\mathcal{H}_{\text{TFD}}$, is a direct product of two state vector spaces, one for the physical, non-tilde operators, $\mathcal{H}$, and the other for tilde ones, $\tilde{\mathcal{H}}$:

$$\mathcal{H}_{\text{TFD}} = \mathcal{H} \otimes \tilde{\mathcal{H}}. \tag{2.1}$$

For a given state $|T\rangle$ in $\mathcal{H}_{\text{TFD}}$, the characteristic function (CF) is

$$\text{CF}(p, q, \tilde{p}, \tilde{q}) \equiv \langle T| \exp[-\mathrm{i}(pP + qQ + \tilde{p}\tilde{P} + \tilde{q}\tilde{Q})]|T\rangle \tag{2.2}$$

where P and Q are momentum and coordinate operators, respectively, and $\tilde{P}$ and $\tilde{Q}$ are their tilde conjugates. The lower-case letters p, q, $\tilde{p}$ and $\tilde{q}$ stand for arbitrary real c-numbers independent of each other. The CF in (2.2) specifies the state $|T\rangle$ up to a trivial overall phase factor.

The observables associated with physical dynamical quantities are given by expectation values of non-tilde operators in TFD. (Note, however, that when one tries to include also observable thermal effects, the expectation values of the cross-product of tilde and non-tilde operators appear as well [15].) As long as we are concerned with such physical dynamical quantities, we only have to deal with a particular CF, $\text{CF}(p, q, \tilde{p} = 0, \tilde{q} = 0)$.

Suppose now that we are given in TFD a two-particle system with dynamical variables P_i, Q_i, $\tilde{P}_i$, $\tilde{Q}_i$ ($i = 1, 2$). We also assume that the two subsystems 1 and 2

are prepared independently, which implies that the state of the total system defined in $\mathcal{H}_{\text{TFD}}$ is factorized, i.e.

$$|T\rangle = |T_1\rangle|T_2\rangle. \tag{2.3}$$

Let us define the centre-of-mass and relative variables as usual:

$$
\begin{aligned}
P_{\text{c}} &= P_1 + P_2 \\
Q_{\text{c}} &= \mu_1 Q_1 + \mu_2 Q_2 \\
P_{\text{r}} &= \mu_1 P_2 - \mu_2 P_1 \\
Q_{\text{r}} &= Q_2 - Q_1
\end{aligned}
\tag{2.4}
$$

and similarly for the tilde operators, where

$$\mu_i = \frac{m_i}{m_1 + m_2} \tag{2.5}$$

and m_i $(i = 1, 2)$ is the mass of the ith particle. Define also

$$G_i = p_i P_i + q_i Q_i \qquad (i = 1, 2, \text{c}, \text{r}). \tag{2.6}$$

We then have the following identity:

$$G_1 + G_2 = G_{\text{c}} + G_{\text{r}} \tag{2.7}$$

where the relations among the c-number parameters are given by

$$
\begin{aligned}
p_{\text{c}} &= \mu_1 p_1 + \mu_2 p_2 \\
q_{\text{c}} &= q_1 + q_2 \\
p_{\text{r}} &= p_2 - p_1 \\
q_{\text{r}} &= \mu_1 q_2 - \mu_2 q_1 .
\end{aligned}
\tag{2.8}
$$

Similar relations apply to the tilde conjugates of equations (2.6).

For the system described above, we consider a situation in which the CF is Gaussian with respect to the centre-of-mass physical variables while nothing is known about the relative coordinates, and ask what can we say about the states of 1 and 2. A similar question was raised in the context of the C^*-algebra formalism by Emch and Hegerfeldt [4] who employed in their proofs a quantum version of Cramér's theorem. However, we will see that it is sufficient to apply the original (classical) Cramér's theorem.

Our assumption that the CF of the centre of mass is a Gaussian is tantamount to writing

$$
\begin{aligned}
\text{CF}_{\text{c}} &= \langle T | e^{-iG_c} | T \rangle \\
&= e^{-\langle G_c^2 \rangle / 2}
\end{aligned}
\tag{2.9}
$$

where $|T\rangle$ and G_c are defined in (2.3) and (2.6) and

$$\langle \cdots \rangle = \langle T| \cdots |T\rangle. \tag{2.10}$$

(Note that we have suppressed here the linear terms $p_c\langle P_c\rangle$, etc. These can be absorbed in a redefinition of the operators as deviations from their mean values, namely P_c should be replaced by $P_c - \langle P_c\rangle$, etc.) CF_c is a general Gaussian in p_c and q_c, i.e. the CM state is a TSS.

Let us introduce a new real variable, u, by the transformation

$$p_i \longrightarrow up_i$$
$$q_i \longrightarrow uq_i$$
$$\tilde{p}_i \longrightarrow u\tilde{p}_i$$
$$\tilde{q}_i \longrightarrow u\tilde{q}_i \qquad (i = 1, 2, c, r). \tag{2.11}$$

Then the condition imposed in the preceding paragraph leads to the form of CF_c where G_c should be replaced by uG_c,

$$\mathrm{CF}_c(u) = \mathrm{e}^{-u^2\langle G_c^2\rangle/2}. \tag{2.12}$$

On the other hand, when

$$p_r = q_r = \tilde{p}_r = \tilde{q}_r = 0 \tag{2.13}$$

equation (2.7) becomes

$$uG_c = uG_1 + uG_2 \tag{2.14}$$

so we have from (2.3)

$$\mathrm{CF}_c(u) = \mathrm{CF}_1(u)\mathrm{CF}_2(u) \tag{2.15}$$

with

$$\mathrm{CF}_i(u) = \langle T_i|\mathrm{e}^{-\mathrm{i}uG_i}|T_i\rangle \qquad (i = 1, 2). \tag{2.16}$$

Furthermore, we may use the following orthonormalized complete set $\{|g_i\rangle|\tilde{g'}_j\rangle\}$ $(i, j = 1, 2)$ associated with the hermitian operators G_i and $\tilde{G}_i$:

$$G_i|g_i\rangle = g_i|g_i\rangle$$
$$\langle g_i|g'_i\rangle = \delta(g_i - g'_i) \qquad (i = 1, 2) \tag{2.17}$$
$$\int_{-\infty}^{\infty} \mathrm{d}g_i \, |g_i\rangle\langle g_i| = 1$$

and similar equations for the tilde variables. Then equation (2.16) can be rewritten as

$$\mathrm{CF}_i(u) = \int \mathrm{d}g_i \, \rho_i(g_i)\mathrm{e}^{-\mathrm{i}ug_i} \tag{2.18}$$

where

$$\rho_i(g_i) \equiv \int \mathrm{d}\tilde{g}_i' \, |\langle T_i|g_i, \tilde{g}_i'\rangle|^2. \tag{2.19}$$

Equations (2.12), (2.15) and (2.18) lead to

$$\int \mathrm{d}g_1 \, \mathrm{d}g_2 \, \rho_1(g_1)\rho_2(g_2)\mathrm{e}^{-\mathrm{i}u(g_1+g_2)} = \int \mathrm{d}g \, \rho_\mathrm{c}(g)\mathrm{e}^{-\mathrm{i}ug} \tag{2.20}$$

where ρ_c is defined by

$$\rho_\mathrm{c}(g) = \int \frac{\mathrm{d}u}{2\pi} \, \mathrm{e}^{-u^2\langle G_\mathrm{c}^2\rangle/2} \mathrm{e}^{\mathrm{i}ug}. \tag{2.21}$$

It is easy to confirm the following properties of ρ_i $(i = 1, 2, \mathrm{c})$: (a) ρ_i is real and non-negative,

$$\rho_i(g) \geq 0 \tag{2.22}$$

and (b) ρ_i is normalized, i.e.

$$\int \mathrm{d}g \, \rho_i(g) = 1. \tag{2.23}$$

Properties (a) and (b) ensure that ρ_i $(i = 1, 2, \mathrm{c})$ are probability distributions. So the classical Cramér's theorem [11] is applicable to the present case: it states that if (2.20) is satisfied with ρ_c defined by (2.21) then $\mathrm{CF}_i(u)$ $(i = 1, 2)$ must also be Gaussian distributions, i.e.

$$\mathrm{CF}_i(u) = \mathrm{e}^{-u^2\langle G_i^2\rangle/2}. \tag{2.24}$$

Therefore each CF_i is Gaussian in p_i, q_i, which means that each $|T_i\rangle$ is a TSS.

Summarizing this section, we have reached the conclusion that when two systems, 1 and 2, are prepared independently and the CF of their centre-of-mass variables is Gaussian then each CF_i $(i = 1, 2)$ must be Gaussian. This result provides an alternative proof of the quantum Cramér's theorem. We would like to note that we found it very convenient to formulate the proof within TFD, but it could also be done within the usual formalism of quantum statistical mechanics.

3. Separability of the centre-of-mass and relative coordinates—bifactorizable functions

Within the formalism of TFD one considers a mixed state (in the non-tilde, i.e. physical space) as a pure state in the expanded space. Thus an arbitrary density matrix ρ of system 1 (in the physical space) can be expressed through a single wavefunction in the TFD space, i.e.

$$\rho(1) \longrightarrow |\psi(1, \tilde{1})\rangle\langle\psi(1, \tilde{1})| \tag{3.1}$$

Relating this to the considerations in the previous section, we now study the following problem: given (i) that two particles 1 and 2 were prepared independently, i.e.

$$\rho(1, 2) = \rho_1(1)\rho_2(2) \tag{3.2}$$

and (ii) their TFD CM state (obtained by tracing over the relative coordinates in TFD space) is a pure state (in TFD space), what can we say about the states of particles 1 and 2? Two points are worth mentioning before we proceed. First, here we do not require the CM to be a Gaussian, in contrast to the case in the previous section dealing with the quantum version of Cramér's theorem; rather we require the CM to be in a pure state in the expanded space. This brings us to the second point: a density matrix pertaining to the CM can always be represented as a pure state in the CM TFD space, *but* that does not mean that tracing over the relative coordinates necessarily yields a pure state. In fact, the information that the tracing yields a pure state has physical meaning, as we shall show. Indeed, the two assumptions imply that the combined density matrix factorizes not only in the individual particles' coordinates but also in the center-of-mass and relative coordinates, i.e.

$$\rho_1(1)\rho_2(2) = \rho_c(R)\rho_r(r) \tag{3.3}$$

and, furthermore, it follows that all the density matrices involved are Gaussian, i.e.

$$\langle x|\rho|x'\rangle = N \exp(-ax^2 - a^*x'^2 + 2cxx' + bx + b^*x') \tag{3.4}$$

where x stands for any of the four coordinates 1, 2, R, r, N is a normalization constant, and c is non-negative. For this case we obtain a stronger version of the result of the previous section, in as much as the Gaussian form was not assumed but, rather, obtained.

The validity of equation (3.3) follows from a theorem by von Neumann [16]. This theorem tells us that if we take the trace with respect to the relative coordinates of a two-particle density matrix and obtain a pure state density matrix for the CM, then the two-particle density matrix is a product of a pure state CM density matrix and a relative coordinates density matrix. In the present case the 'pure state' CM density matrix is 'pure' in the expanded space. This confirms equation (3.3), which implies that the density matrix for the combined systems is factorizable both as single-particle density matrices and in the CM and relative coordinates (cf equation (3.3)). Thus the density matrix for the whole system possesses the bifactorizability property [12]. In [12] it is shown (cf equations (20) and (21)) that this implies that all density matrices involved are of the Gaussian form equation (3.4), and the constants a and c are common to all the four density matrices of equation (3.3).

As was pointed out in the introduction, these states can be viewed as TSS. Hence the equality of the constants a and c may be interpreted physically as implying that all the states have a common temperature. This corresponds to corollary 3.5 of EH.

4. Summary

In this paper we have dealt with two independently prepared particles and considered the following two cases: (1) tracing out the relative coordinates yields a pure state

density matrix (in TFD) for the centre of mass, and (2) tracing out the relative coorinates does not yield a pure state (in TFD) for the CM; however, it is given, in analogy to the case considered by EH, that the physical part of the CM density matrix is a Gaussian (i.e. a TSS). In the first case we showed that the total system is also separable in the CM and relative coordinates and that all four density matrices involved are thermal squeezed states. In the second case we showed that again the two particles were prepared in thermal squeezed states, thereby providing an alternative proof of the quantum version of Cramér's theorem [4].

Acknowledgments

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